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Once-punctured Klein bottles in knot complements ☆

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Abstract

We find the family of all knots in S^3 which are spanned by two essential once-punctured Klein bottles with boundary slopes at distance 4, thus settling a conjecture by K. Ichihara, M. Ohtouge, and M. Teragaito. We also address the more general question of when a knot exterior in an arbitrary 3-manifold contains two essential once-punctured Klein bottles with distinct boundary slopes.

MSC: 57M25; 57N10

Keywords: Once-punctured Klein bottles; Seifert Klein bottles; Knot complements

1. Introduction

Let K be a knot in S^3 with exterior $X_K = S^3 \setminus \operatorname{int} N(K)$. We will say that a once-punctured Klein bottle P properly embedded in X_K is a *Seifert Klein bottle* for K if P has integral boundary slope. By [3,5], if K is nontrivial and not a 2-cable, a once-punctured Klein bottle in X_K is *essential* (i.e., geometrically incompressible and boundary-incompressible) and Seifert with boundary slope an integral multiple of 4; moreover, if X_K contains two Seifert Klein bottles with different boundary slopes, then their slopes are at a distance 4 or 8 from each other, and if at distance 8 then K is the pretzel knot $P(2,1,1)$, the figure-8 knot.

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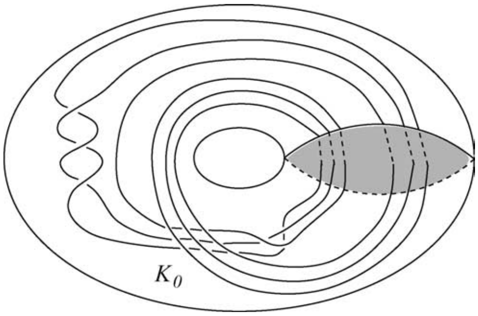


Fig. 1. A Klein pair.

Regarding the case of a knot $K \subset S^3$ that has two essential Seifert Klein bottles with boundary slopes at distance 4, Ichihara, Ohtouge, and Teragaito conjectured in [5, §3] that K must belong to a certain family of knots. In this paper we establish this conjecture by analyzing the possible graphs of intersection between P and Q, which can be determined via the *parity rule* of [5]. In fact, we show that the conjecture holds for knots in a large family of 3-manifolds.

For the general case when P, Q are essential once-punctured Klein bottles properly embedded in a 3-manifold with torus boundary and with boundary slopes at distance Δ, which is necessarily even, we show that $\Delta \leqslant 8$ and use results of Gordon [2] to determine the ambient 3-manifold for $\Delta = 6$ or 8; we also find information about the ambient 3-manifold for $\Delta = 2, 4$.

We work in the PL-category. Any 3-manifold is assumed to be compact and orientable, and any surface in a 3-manifold is assumed to be properly embedded, unless otherwise stated. In order to state our first result, we will need the following definition. A *Klein pair* (V, K_0) consists of a solid torus V and a knot $K_0 \subset \operatorname{int} V$ such that (V, K_0) is homeomorphic (as a pair) to the pair in Fig. 1. Notice that the pair (V, K_0) shown in Fig. 1 is the same pair of Fig. 8 in [5], so K_0 represents the Fintushel–Stern pretzel knot $P(-2, 3, 7)$ in S^3 under the given embedding of (V, K_0). It is shown in [5, Theorem 3] (see also Remark 3.4) that K_0 bounds two Seifert Klein bottles in V with boundary slopes at distance 4 and that K_0 is not a 2-cable knot, so any Seifert Klein bottle for K_0 in S^3 is essential. In this context, the conjecture in [5, §3] states that *if K is a knot in S^3 spanned by two essential Seifert Klein bottles with boundary slopes at distance 4, then K must be part of some Klein pair embedded in S^3*. The following theorem settles this conjecture affirmatively.

Theorem 1.1. *A nontrivial knot $K \subset S^3$ is spanned by two essential Seifert Klein bottles with boundary slopes at distance* 4 *iff K is part of some Klein pair embedded in S^3.*

This theorem follows directly from the results of Section 2, Proposition 3.3, and [5, Theorem 3]. It is also a corollary of the more general Proposition 5.3, where embeddings of Klein pairs in arbitrary 3-manifolds are considered.

Theorem 1.2 below summarizes our results on the question of when an arbitrary knot exterior contains essential once punctured Klein bottles with distinct boundary slopes; we will need the following definitions. If α_1, α_2 are slopes in a closed torus T at distance $\Delta = \Delta(\alpha_1, \alpha_2)$, define for $\{i, j\} = \{1, 2\}$ the number

$$d_i = \min\bigl\{\Delta(\alpha_i, \mu) \mid \mu \text{ a slope in } T \text{ with } \Delta(\alpha_j, \mu) = 1\bigr\}$$

and set $d = d(\alpha_1, \alpha_2) = \min\{d_1, d_2\}$; this is the same parameter defined in [2, §2]. We will say that a slope μ in T *realizes* $d(\alpha_1, \alpha_2)$ if $\Delta(\alpha_i, \mu) = 1$ and $\Delta(\alpha_j, \mu) = d(\alpha_1, \alpha_2)$ for some $\{i, j\} = \{1, 2\}$. Notice that such a μ is unique if $\Delta \geqslant 3$ and $d = 1$; also, $d = 1$ for $\Delta = 2, 3, 4, 6$, while $d = 1$ or 3 for $\Delta = 8$. If N^3 is a 3-manifold with distinct torus boundary components T_i, $i = 1, \ldots, s$, and r_i is a slope in T_i for each i, we will denote by $N^3(r_1, \ldots, r_s)$ the manifold obtained by Dehn-filling N^3 along the slopes r_i, and by K_{r_i} the core of the Dehn-filling solid torus along T_i. Let $\mathcal{W}$ denote the exterior of the Whitehead link, with $\mathcal{W}(r)$ as in [2].

Theorem 1.2. *Let M^3 be an irreducible* 3*-manifold with an incompressible torus boundary component T. If (M^3, T) contains two essential once-punctured Klein bottles with boundary slopes r and s, then $\Delta(r, s) \leqslant 8$, and we have the following possibilities; let $\Delta = \Delta(r, s)$, $d = d(r, s)$, and μ any slope realizing $d(r, s)$:*

- (i) *if $\Delta = 2$ then $M^3(\mu)$ contains a closed nonorientable surface of genus* 2 *or* 3 *which intersects K_μ in* 0 *or* 2 *points;*
- (ii) *if $\Delta = 4$ then either $M^3(\mu)$ contains a closed Klein bottle which intersects K_μ in* 0 *or* 4 *points, or K_μ is part of some Klein pair embedded in $M^3(\mu)$ and (M^3, T) does not contain any Möbius bands;*
- (iii) *if $\Delta = 6$ then $M^3 = \mathcal{W}(2)$;*
- (iv) *if $\Delta = 8$ and $d = 1$ then $M^3 = \mathcal{W}(1)$ (the figure-8 knot exterior);*
- (v) *if $\Delta = 8$ and $d = 3$ then $M^3 = \mathcal{W}(-5)$.*

In particular, the essential punctured tori in $\mathcal{W}(1), \mathcal{W}(2), \mathcal{W}(-5)$ with boundary slopes at maximal distance are the frontiers of regular neighborhoods of essential π_1-injective once-punctured Klein bottles, and only (ii) *and* (iv) *can be realized by knot exteriors in integral homology* 3*-spheres.*

Parts (iii)–(v) in Theorem 1.2 essentially follow from [2], while (ii) is the content of Proposition 5.3; the bound $\Delta \leqslant 8$ and parts (iv), (v) generalize [5, Lemma 12, Proposition 13].

The following corollary to the results of Section 4 gives a natural reason for excluding 2-cable knots in S^3 (which include 2-torus knots) from any discussion involving essential once-punctured Klein bottles.

Corollary 1.3. *Let K be a nontrivial 2-cable knot in S^3 with exterior X_K. Then any once-punctured Klein bottle in X_K is boundary compressible, hence it is obtained by adding a band along ∂X_K to a Möbius band in X_K.*

Since by [5] the only hyperbolic knots in S^3 which are part of some Klein pair are the pretzel knot $P(-2, 3, 7)$ and its reflection, the above theorems allow us to determine which hyperbolic knots admit more than one Seifert Klein bottle boundary slope; recall that, by [5], any knot in S^3 which is nontrivial and not a 2-cable admits at most two such slopes.

Corollary 1.4. *The only hyperbolic knots in S^3 with more than one Seifert Klein bottle boundary slope are the pretzel knots $P(2,1,1)$ and $P(\mp 2, \pm 3, \pm 7)$.*

The paper is organized as follows. In Section 2 we classify the graphs of intersection of two once-punctured Klein bottles P, Q with $\Delta(\partial P, \partial Q) = 4$ and show that, under some constraints, such graphs may be assumed to be of a certain type we call *minimal*. Section 3 contains the topological analysis of minimal graphs of intersection and the proof of Proposition 3.3, which along with [5, Theorem 3] completes the proof of Theorem 1.1. In Section 5 we consider the more general problem of when an embedding of a Klein pair in a 3-manifold yields essential Seifert Klein bottles with boundary slopes at distance 4 and give the proof of Proposition 5.3, which generalizes Theorem 1.1. The proof of Proposition 5.3 relies on some special properties of the 6-string braid K in a Klein pair (V, K) which are developed in Section 4; this section also contains the proofs of the bound $\Delta \leqslant 8$ of Theorem 1.2 and Corollary 1.3. Finally, the proof of Theorem 1.2 is completed in Section 6.

2. Graphs of intersection for $\Delta = 4$

In this section we show that the graphs of intersection between two essential Seifert Klein bottles of a knot in S^3 with boundary slopes at distance 4 almost are combinatorially unique. We will work in a slightly more general setting.

Let M^3 be an irreducible 3-manifold with an incompressible torus boundary component T. Let P, Q denote essential once-punctured Klein bottles in (M^3, T), with boundary slopes r, s in T, respectively, at distance 4. Let μ be the unique slope in T realizing $d(r,s) = 1$. If $K = K_\mu \subset M^3(\mu)$, then we can refer to P, Q as *Seifert Klein bottles* for K as before.

Since M^3 is orientable, any regular neighborhood $N(P)$ of P in M^3 is a genus two handlebody, and the pair $(N(P), \partial P)$ is homeomorphic to the model in Fig. 2. We will always use this model of $(N(P), \partial P)$, for any such P, in the sequel.

We may assume that P and Q have been isotoped so as to intersect transversely and minimally, so ∂P and ∂Q intersect transversely in 4 points and any circle component of

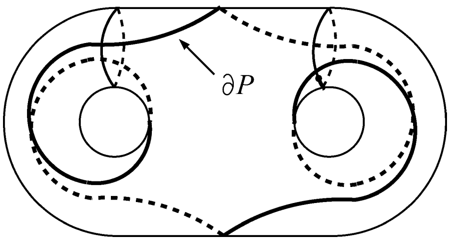


Fig. 2. The regular neighborhood $N(P) \subset M^3$.

$P \cap Q$ is essential in both P and Q. Following [5], we orient ∂P and ∂Q in the same direction with respect to μ, and proceed to number the points $\partial P \cap \partial Q$ with the labels 1, 2, 3, 4 in their order of appearance along ∂P; since $d(r,s) = 1$, the points $\partial P \cap \partial Q$ appear in the same order along ∂Q (cf. [5, §3]).

Let $G_P = P \cap Q \subset P$ and $G_Q = P \cap Q \subset Q$ be the graphs of intersection between P and Q. The main result of this section, Lemma 2.4, classifies the combinatorics of these two graphs, under certain constraints; for the case $M^3(\mu) = S^3$, the result is mentioned in [5, §3].

Before discussing the combinatorics of these graphs of intersection, we state some useful facts about circles and properly embedded arcs in a Klein bottle. Let F denote a once-punctured Klein bottle, so $\widehat{F} = F \cup_\partial$ disk is a closed Klein bottle. Then,

- there are exactly four isotopy classes of essential embedded circles in $\widehat{F}$: the two *centers* of the Möbius bands, the *meridian* (which cuts $\widehat{F}$ into an annulus), and the *longitude* (which separates $\widehat{F}$ into Möbius bands); only the meridian circle is unique in F up to isotopy;
- there are two types of essential arcs properly embedded in F, *positive* and *negative*, as described in [5, §3]. Observe that an arc is positive iff it closes into a meridian or a longitude circle in $\widehat{F}$, and negative iff it closes into a center circle in $\widehat{F}$; in particular, any two disjoint positive arcs in F are parallel.

Lemma 2.1. *The graph pair G_P, G_Q is combinatorially isomorphic* (*labels included*) *to one of the graph pairs* (a), (b), (c), (e), (f) *in Fig.* 3. *In particular, $P \cap Q$ may only have circles of intersection which are longitudes in both P and Q, or centers in both P and Q, with at most one center circle of intersection.*

Proof. Since $\Delta(\partial P, \partial Q) = 4$, each of the graphs G_P, G_Q consists of two arcs and, perhaps, some circle components. Using the *parity rule* of [5], it is not hard to see that the possible graph pairs G_P, G_Q are the ones listed in Fig. 3; here, the graph pairs (e) and (f) have the same arc combinatorics, but only (f) is assumed to contain circle components. The graph pair (d) has to be excluded, as the correct labelling of the intersection points is not possible.

In the cases (a), (b), (c) of Fig. 3, since the arcs of at least one graph cut their corresponding surface into a disk, there can be no circle components in $P \cap Q$. Regarding circle intersections in case (f), as M^3 is orientable, if α is a circle component of $P \cap Q$ then either α preserves orientation in both P and Q or it reverses orientation in both P and Q. Since in this case $P \cap Q$ can have no meridian circles of intersection, any circle component of $P \cap Q$ is either a longitude in both P and Q, or a center in both P and Q, and there can be at most one center component in $P \cap Q$. $\square$

Recall that $M^3(\mu) = M^3 \cup_T N(K)$, where $K = K_\mu$ and $N(K)$ is the solid torus in the Dehn-filling $M^3(\mu)$. We now derive some topological properties of the manifolds $M^3, M^3(\mu)$ from the graphs G_P, G_Q.

Lemma 2.2. *If the graph pair G_P, G_Q is one of the pairs in Fig.* 3 (a)–(c), (e), *then $M^3(\mu)$ contains a closed Klein bottle which intersects K_μ in* 4 *points. In particular, M^3 contains a closed nonorientable surface.*

Proof. We proceed to analyze each such graph pair.

Case 1. Fig. 3(a).

Let $N(P)$ be a small regular neighborhood of P in M^3, so that $Q \setminus \operatorname{int} N(P)$ consists of a disk D_Q. Let $A \subset \partial T$ denote the annulus $T \cap N(P)$ with core ∂P, and A_K the complementary annulus $T \setminus \operatorname{int} A$.

Since μ realizes $d(r,s) = 1$, the circles $\partial P, \partial Q$ may be assumed to lie in T as shown in Fig. 4 (up to homeomorphism); the models correspond to whether the algebraic intersection number $\partial P \cdot \partial Q$ in T is positive or negative.

Clearly $\partial D_Q \subset N(P) \cup_A N(K)$. Therefore, since A and A_K are parallel in $N(K)$, it is possible to extend D_Q via meridian disks of $N(K)$ to a disk D_Q^* properly embedded in $M^3(\mu) \setminus \operatorname{int} N(P)$, with $\partial D_Q^* \subset \partial N(P)$. Fig. 5 shows the only two possible combinatorics for the circle $\partial D_Q^* \subset \partial N(P)$; notice the two parallel meridian disks of $N(P)$, which are generated by the arcs of G_P under the I-bundle structure of $N(P)$. As before, the possibilities correspond to whether $\partial P \cdot \partial Q$ is positive or negative. We will concentrate in the case of Fig. 5(a), since the other case can be treated along the same lines and yields the same conclusion.

Observe that, with the embedding of $N(P)$ in S^3 as given in Fig. 5(a), the circle ∂D_Q^* represents the pretzel knot $P(-1,2,-1)$ ($P(-1,-2,-1)$ in the case of Fig. 5(b)), hence it bounds a once-punctured Klein bottle P' in $N(P)$. It follows that $P' \cup D_Q^*$ is a closed Klein bottle in $M^3(\mu)$ which intersects K_μ in $\Delta = |\partial D_Q^* \cap \partial P| = 4$ points.

Remark 2.3. We shall continue to use the above construction whereby, for example, the disk D_Q is extended to a disk D_Q^* which is properly embedded in $M^3(\mu) \setminus \operatorname{int} N(P)$, without further ado. Also, there will always be two possible realizations in $\partial N(P)$ of a circle like ∂D_Q^*, and we shall work with only one of them when no loss of generality is involved.

Case 2. Fig. 3 (b) and (c).

These two cases are similar to case 1: with the same notation as above, the circle ∂D_Q^* bounds a once-punctured Klein bottle in $N(P)$. A representation of ∂D_Q^* is given in Fig. 6(b) and 7(b), which correspond to the cases of Fig. 3(b) and (c), respectively. In Fig. 3(c) we have assumed that $a = 1$ and $b = 3$, the other assignment for a and b can be dealt with in a similar fashion.

Case 3. Fig. 3(e).

Let $N(P)$ be a small regular neighborhood of P in M^3. Then $Q \setminus \operatorname{int} N(P)$ consists of a disk D_Q and a Möbius band M_Q (see Fig. 8(a)). As usual, extend D_Q and M_Q

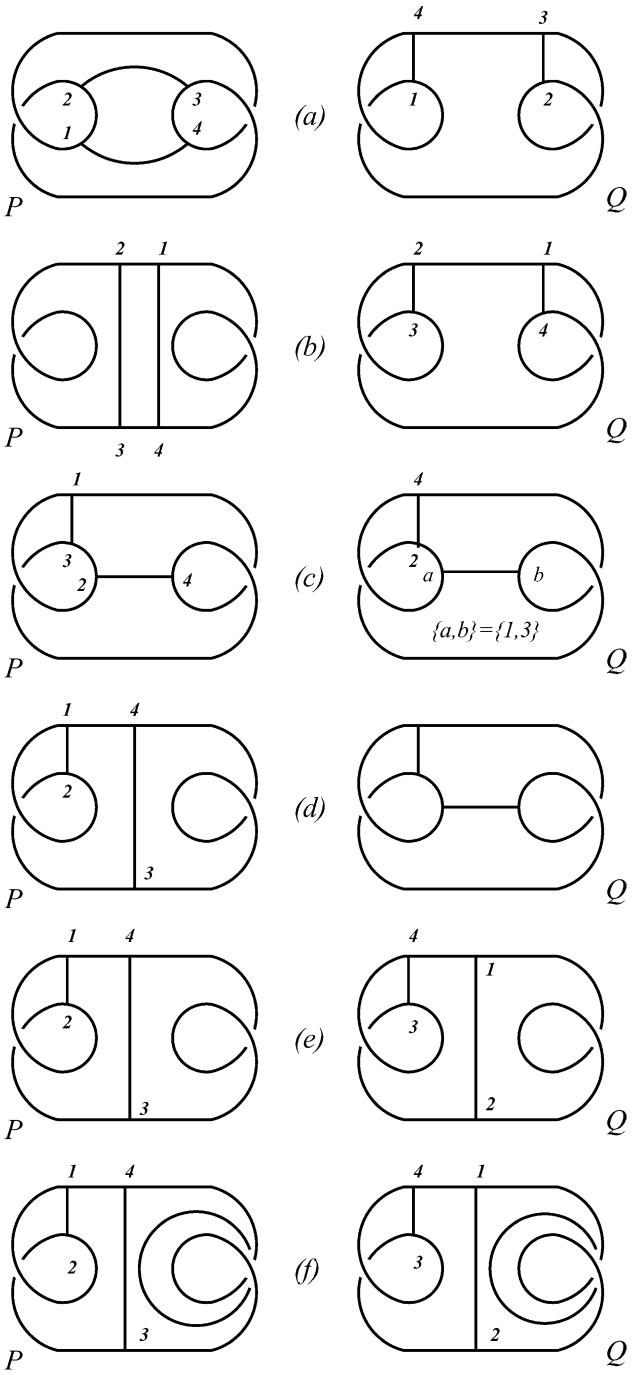


Fig. 3. The graph pairs G_P, G_Q for $\Delta = 4$.

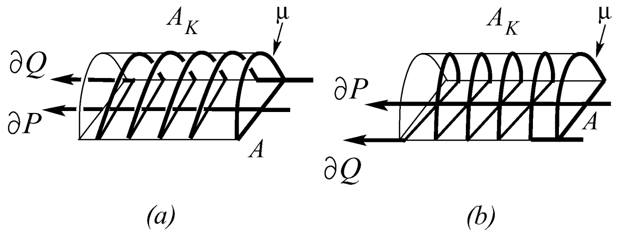


Fig. 4. The circles ∂P, ∂Q, μ in T.

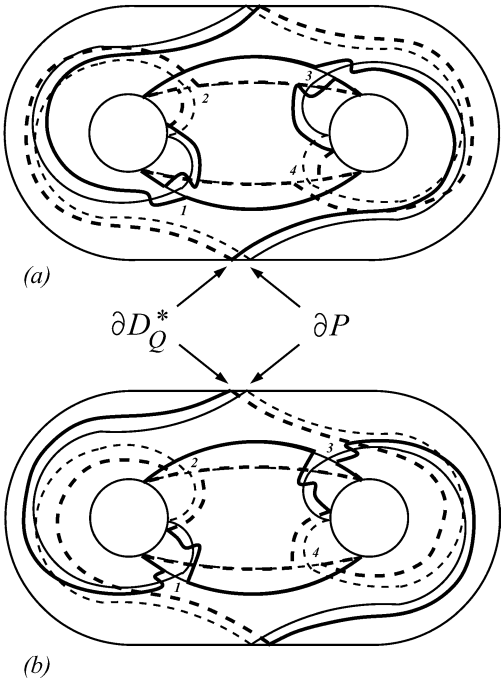
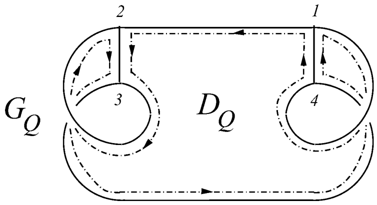
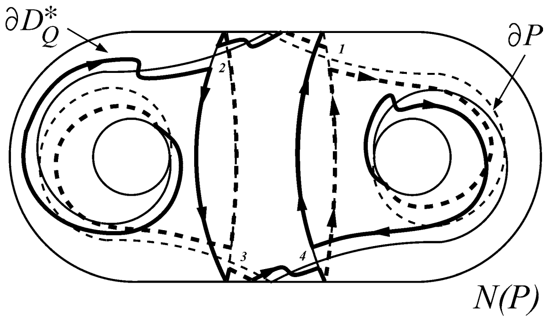


Fig. 5.

to D_Q^* and M_Q^*, respectively, which lie properly embedded in $M^3(\mu) \setminus \operatorname{int} N(P)$. One realization of the circles $\partial D_Q^*, \partial M_Q^* \subset \partial N(P)$ is shown in Fig. 8(b), (c). Taking the circles x, y shown in Fig. 8(c) as a basis for $\pi_1(N(P))$, so that $\pi_1(N(P)) = \langle x, y \mid - \rangle$, we see that ∂M_Q^* and ∂D_Q^* represent x and xy^2, respectively, in $\pi_1(N(P))$. Hence $\mathcal{V} = N(P) \cup$ (2-handle along D_Q^*) is a solid torus in M^3 with M_Q^* properly embedded in $M^3(\mu) \setminus \operatorname{int} \mathcal{V}$. Moreover, $\pi_1(\mathcal{V}) = \langle x, y \mid xy^2 = 1 \rangle = \langle y \mid - \rangle$, so ∂M_Q^* represents y^2 in $\pi_1(\mathcal{V})$. Therefore, ∂M_Q^* bounds a Möbius band M' in $\mathcal{V}$, and hence $M_Q^* \cup_\partial M'$ is a closed Klein bottle in $M^3(\mu)$.

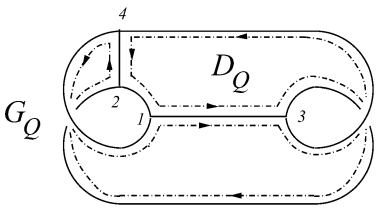


(a)

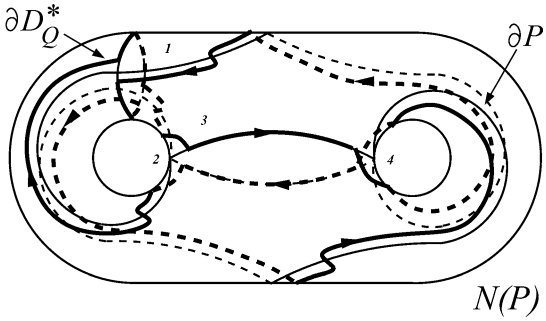


(b)

Fig. 6.



(a)



(b)

Fig. 7.

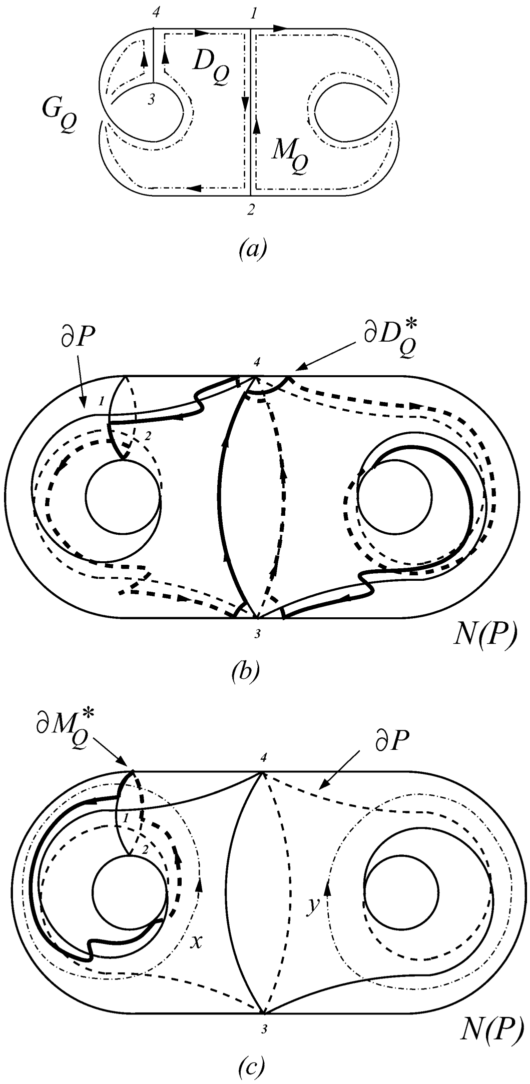


Fig. 8.

This last case completes the proof of the lemma, after noticing that in all cases the given closed Klein bottle in $M^3(\mu)$ intersects the knot $K = K_\mu$ in $\Delta = 4$ points, so M^3 itself contains a closed nonorientable surface. $\square$

We now show that in some cases, including the case of interest $M^3(\mu) = S^3$, we may assume that the graph pair G_P, G_Q is that of Fig. 3(f) with exactly one circle component, which is a center in both P and Q. We call such a graph pair *minimal*.

Lemma 2.4. *Either $M^3(\mu)$ contains a closed Klein bottle that intersects $K = K_\mu$ in* 0 *or* 4 *points, or there is a once-punctured Klein bottle Q' in (M^3, T) with $\partial Q' = \partial Q$ which intersects P transversely in a minimal graph pair.*

Proof. By Lemma 2.1, the graph pair G_P, G_Q must correspond to one of the pairs in Fig. 3(a)–(c), (e), (f). In the cases of (a)–(c),(e) there is a closed Klein bottle in $M^3(\mu)$ intersecting K in 4 points by Lemma 2.2. Consider now the case of (f), that is, $P \cap Q$ has circle components, each of which is either a longitude or a center of P, Q by Lemma 2.1.

Case 1. Each circle component of $P \cap Q$ is a longitude.

Let l be an innermost circle component of $P \cap Q$ in P; that is, l bounds a Möbius band M_P in P with $Q \cap \operatorname{int} M_P = \emptyset$. Then l also bounds a Möbius band M_Q in Q, and $\operatorname{int} M_P \cap \operatorname{int} M_Q = \emptyset$. Hence $M_P \cup_l M_Q$ is a closed Klein bottle in $M^3 \subset M^3(\mu)$ which intersects K_μ in 0 points.

Case 2. $P \cap Q$ has a center circle component.

Let $M_Q \subset Q$ be the closure of the Möbius band component of $Q \setminus$ (arcs of G_Q). We claim that if $P \cap Q$ also contains longitude circle components, then it is possible to replace Q by a new once-punctured Klein bottle Q' in (M^3, T), such that

(a) $Q' \cap Q = Q \setminus \operatorname{int} M'$ *for some Möbius band* $M' \subset \operatorname{int} M_Q$,
(b) $P \cap Q'$ *has one center circle component but fewer longitude circle components than* $P \cap Q$, *and*
(c) *the graph pair $P \cap Q' \subset P$ and $P \cap Q' \subset Q'$ is still equivalent to that of Fig.* 3(f).

In general, given any Q as above, if Q' satisfies conditions (a), (b), and (c), we will say that *Q' has been obtained from Q by a Möbius band exchange.*

Let l be an outermost longitude circle of G_P, that is, l bounds a Möbius band M_P in P such that $Q \cap \operatorname{int} M_P$ contains all the circle components of $P \cap Q$ other than l. The circle l also bounds a Möbius band in Q, which we denote by M_l. Let N be a small regular neighborhood of M_P in M^3 (a solid torus) such that $Q \cap \partial N$ has an annulus component $A_l \subset Q$ containing l as a core in its interior; the situation is described in Fig. 9(a), where $M_P \cap M_l$ is assumed to consist of just a center circle component c for simplicity.

Out of the two boundary components $\partial_1 A_l$ and $\partial_2 A_l$ of A_l, only one of them, say $\partial_1 A_l$, lies in $\operatorname{int} M_l$. Let M' be a Möbius band properly embedded in N such that $\partial M' = \partial_2 A_l$ and $M' \cap M_P$ is a center curve in each of M_P, M' (see Fig. 9(b)).

Now let $Q' = (Q \setminus \operatorname{int}(A_l \cup M_l)) \cup_{\partial_2 A_l} M'$. It is not hard to see that Q' satisfies conditions (a)–(c) above, so Q' has been obtained from Q via a Möbius band exchange, and that $P \cap Q'$ contains only one circle component, which is a center. □

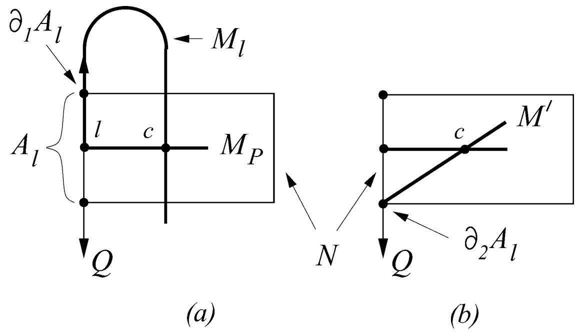


Fig. 9. A Möbius band exchange.

3. Minimal graph pairs

In this section we further study the case when the Seifert Klein bottles P and Q from Section 2 (here not necessarily essential) intersect in a minimal graph pair, and find information about the embedding of $K = K_\mu$ in $M^3(\mu)$.

Let $N(P)$ denote a small regular neighborhood of P in M^3. Then $Q \setminus \operatorname{int} N(P)$ consists of two components, a disk D_Q and an annulus A_Q. We will make use of the meridian disks D_x, D_y of $N(P)$ shown in Fig. 10(b), which form a complete disk system for the handlebody $N(P)$; notice that D_x intersects P in the negative arc of G_P.

Lemma 3.1. *Let D_Q^* denote the extension of D_Q to a disk properly embedded in $M^3(\mu) \setminus \operatorname{int} N(P)$. Then ∂D_Q^* may be assumed to be the circle shown in Fig. 10(b), and $V^* = N(P) \cup (2$-handle along $D_Q^*) \subset M^3(\mu)$ is a solid torus.*

Proof. The circle $\partial D_Q^* \subset \partial N(P)$ can be easily constructed from the combinatorial information of G_P and G_Q; by exchanging the roles of P and Q, if necessary, this circle may be assumed to be the one shown in Fig. 10(b). That V^* is a solid torus follows from the fact that ∂D_Q^* intersects the meridian disk D_x of the handlebody $N(P)$ transversely in one point (which implies that ∂D_Q^* represents a primitive element in $\pi_1 N(P)$). □

Lemma 3.2. *There are disks D_m and D_w in $N(P)$ such that*

(a) *D_x, D_m, and D_w are disjoint,*
(b) *D_m is a non-separating compression disk for $\partial N(P) \setminus \partial D_Q^*$ in $N(P)$, and*
(c) *D_w is a separating compression disk for $\partial N(P) \setminus \partial D_Q^*$ in $N(P)$, with D_x and D_m lying in opposite sides of D_w.*

In particular, D_x and D_m form a complete disk system for $N(P)$.

Proof. Since $|\partial D_Q^* \cap \partial D_x| = 1$ and $|\partial D_Q^* \cap \partial D_y| = 2$, the non-separating compression disk D_m can be constructed from the disk D_y and two copies of the disk D_x by tubing them along some suitable arcs of ∂D_Q^*, while the separating disk D_w is obtained from the

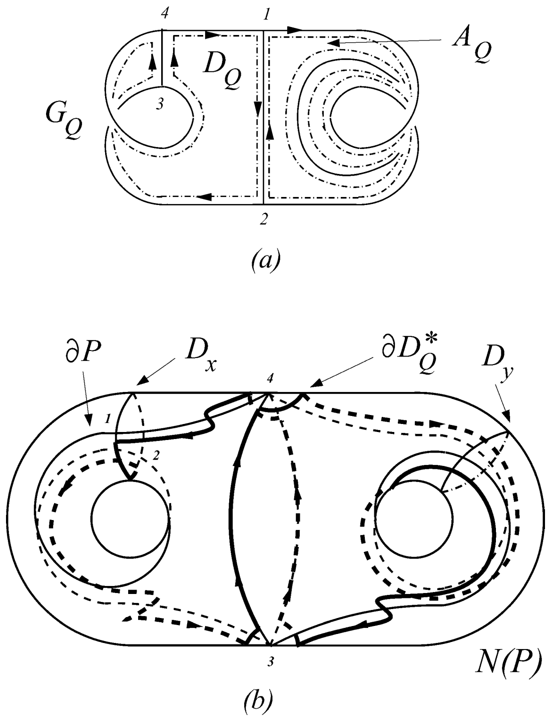


Fig. 10.

frontier of a small regular neighborhood in $N(P)$ of $D_x \cup \partial D_Q^*$. The circles ∂D_m and ∂D_w are shown in Fig. 11. □

It can be proved that the non-separating compression disk D_m for $\partial N(P) \setminus \partial D_Q^*$ is unique up to isotopy in $N(P)$, but we shall not make use of this fact.

Let K^* be an isotopic copy of K obtained by pushing ∂P slightly into the interior of $N(P)$, so that $K^* \subset \operatorname{int} V^*$, where V^* is the solid torus of Lemma 3.1. We now proceed to identify the pair (V^*, K^*), up to homeomorphism.

Proposition 3.3. *The pair $(V^*, K^*) \subset M^3(\mu)$ is a Klein pair.*

Proof. We orient the circles $\partial D_x, \partial D_m, \partial D_w,$ and ∂D_Q^* as indicated in Fig. 11, while $\partial N(P)$ is oriented by its outer pointing normal vector.

Let V_x and V_m be the closures of the components of $N(P) \setminus D_w$; V_x and V_m are solid tori with meridian disks D_x and D_m, respectively. Notice that $D_w = V_x \cap V_m = \partial V_x \cap \partial V_m$ is disjoint from D_x and D_m.

We now homeomorph the solid tori V_x and V_m so as to agree with the models in Fig. 12, each oriented with its outer pointing normal vector. Their union $V_x \cup_{D_w} V_m$ (identified via only full twists along D_w) is now a homeomorph of $N(P)$, which we denote by N'_P.

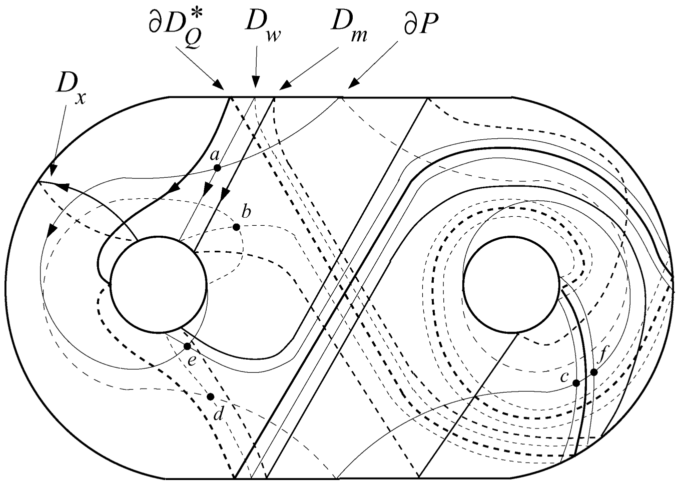


Fig. 11. The disks D_w, D_m in $N(P)$.

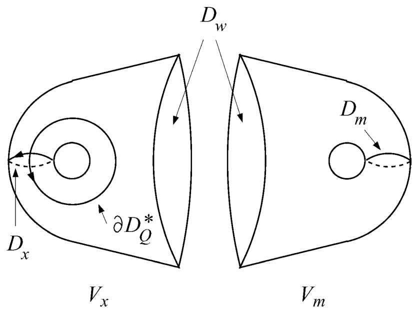


Fig. 12.

We also denote by T_x and T_m the once-punctured tori $\partial V_x \setminus \operatorname{int} D_w$ and $\partial V_m \setminus \operatorname{int} D_w$, respectively.

The circles ∂D_x and ∂D_Q^* will be oriented as in Fig. 12, which agree with their orientations as given in Fig. 11. It remains to determine the correct orientations for the circles ∂D_w and ∂D_m in $\partial N_P'$. To do this, we first isotope each of the disks D_x, D_m in $V_x, V_m \subset N(P)$ so as to intersect D_w in a single arc, and record the orientations of these arcs, as shown in Fig. 13. We can orient ∂D_w in N_P' from the fact that the triple consisting of the vector tangent to ∂D_x at the point A in Fig. 13(a), the vector tangent to ∂D_w at A, and the normal vector of $\partial N(P)$ at A is an orientation frame for $N(P)$, and from here the orientation of ∂D_m in N_P' follows immediately by similar considerations at the point C of Fig. 13(b). The complete set of oriented circles is shown in Fig. 14.

Finally, we show how ∂P is embedded in $\partial N_P'$. First, observe that ∂P intersects ∂D_w in the six points a, b, c, d, e, f of Fig. 11 in the patterns shown in Fig. 15. Thus, $T_x \cap \partial P$ and $T_m \cap \partial P$ each consists of three properly embedded arcs. Since the oriented arcs $T_x \cap \partial P$ have endpoints $a \to e$, $b \to d$, and $c \to f$, it follows that the first two arcs are parallel

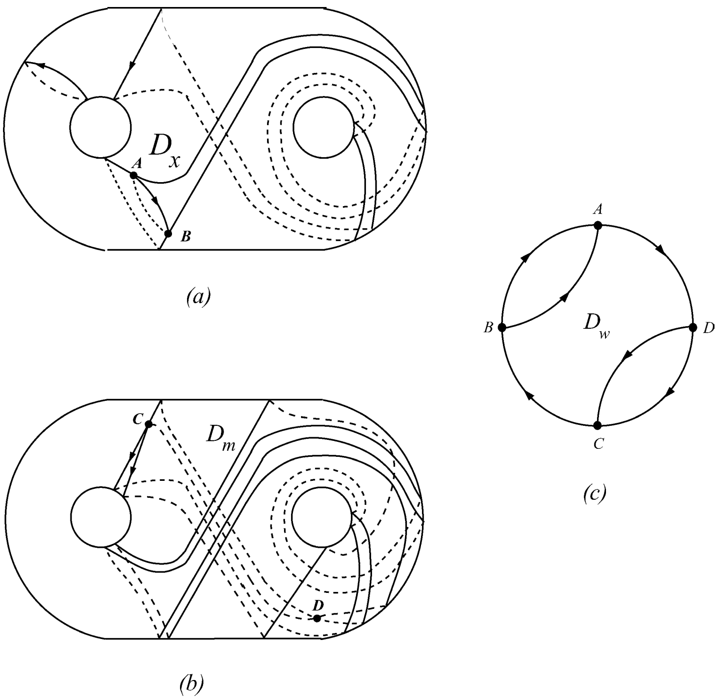


Fig. 13.

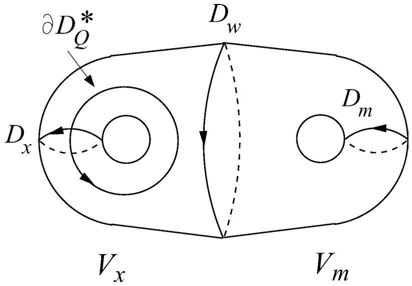


Fig. 14.

in T_x; taking into account the oriented intersections between these arcs and the circles $\partial D_x, \partial D_Q^*$, we can see that their combinatorics are uniquely determined in T_x, as shown in Fig. 16. The three arcs $T_m \cap \partial P$ have endpoints $f \to a$, $e \to b$, and $d \to c$, so the three arcs are mutually parallel in T_m; taking into account their oriented intersections with the circle ∂D_m, we see that their combinatorics are uniquely determined in T_m, up to full twists along ∂D_m, as shown in Fig. 16. Putting V_x and V_m together to recover N_P', and attaching a 2-handle along the circle $\partial D_Q^* \subset T_x$ (shown in Figs. 14 and 16), we can see immediately that (V^*, K^*) is a Klein pair. $\square$

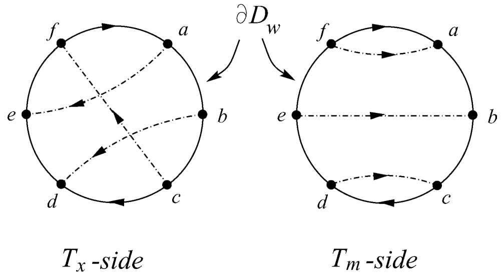


Fig. 15.

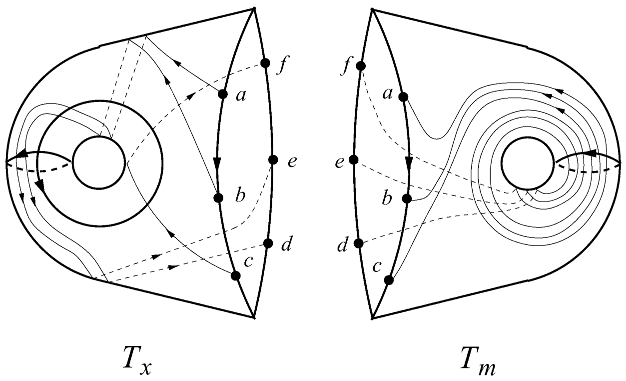


Fig. 16.

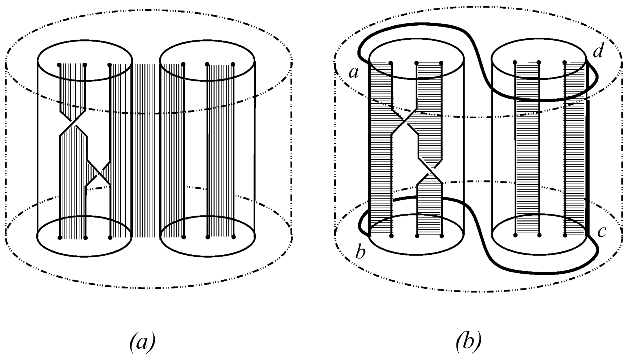


Fig. 17. The standard Seifert Klein bottles of (V, K).

Remark 3.4. For (V, K) a Klein pair, that K indeed bounds two Seifert Klein bottles in V with boundary slopes at distance 4 can be seen directly from Fig. 17, where a braid representation of K is shown (in Fig. 17(b), the rectangle with vertices a, b, c, d is supposed to bound a disk disjoint from the other parts of the figure). As in [5, Theorem 3], the Klein pair is obtained by gluing the ends of the outer tube via an odd number of half-twists. This pair of Seifert Klein bottles is somewhat different from the pair shown in

Fig. 9 of [5], and the intersection $P \cap Q$ is easier to identify. Notice that if the arcs ad and bc in Fig. 17(b) are replaced with similar arcs which run $(2n + 1)$ half-turns around the left inner tube and then $(2n + 1)$ half-turns around the other tube (so that the case shown corresponds to $n = 0$), then $P \cap Q$ will contain exactly $n + 1$ circle components: one center circle and n longitude circles. This example shows that the graph pair G_P, G_Q is, in general, not minimal.

We will call P (Fig. 17(a)) and Q (Fig. 17(b)) the *standard* Seifert Klein bottles for K in V with boundary slopes at distance 4. By [5], these surfaces are essential in the exterior of K in V.

4. Möbius bands and once-punctured Klein bottles

Let (V, K) be a Klein pair. Let W be a solid torus in V whose core is a 2-cable of the core of V and contains K as a 3-string closed braid (see Fig. 17 and [5, proof of Theorem 3]). In this section we establish some properties of the manifolds obtained by surgery on $K \subset W$; some results that may be of independent interest are developed along the way, which lead to simple proofs of Corollary 1.3 and the bound $\Delta \leqslant 8$ in Theorem 1.2.

Let $W_K = W \setminus \operatorname{int} N(K)$ denote the exterior of K in W, and let $\partial_0 W_K$ and $\partial_1 W_K = \partial W$ denote its boundary components, with the meridian slope μ of K in $\partial_0 W_K$. Notice W_K is hyperbolic, being homeomorphic to the exterior of the 2-bridge pretzel link $P(-2, 4, 5)$ in S^3 (cf. [5, Theorem 3]). Let $P' = P \cap W_K$ and $Q' = Q \cap W_K$, where P and Q are the standard Seifert Klein bottles of K in V; P' and Q' are once-punctured Möbius bands in W_K with integral slopes in $\partial_0 W_K$ (relative to μ) at distance 4 and common slope α_W in ∂W running once along W.

Since α_W is integral with respect to W, it follows that $W_K(\mu, \alpha_W) = S^3$. It is not hard to see that the link $K \cup K_{\alpha_W} \subset S^3$ is the one represented in Fig. 18(a); in fact, this link is the pretzel $P(4, -1, -2)$ (the 2-bridge link 6_2^2 in Rolfsen's table), whose components are trivial knots in S^3, and all three links in Fig. 18 are isotopic. Thus, the embedding of K in $S^3 \setminus \operatorname{int} N(K_{\alpha_W}) \approx S^1 \times D^2$ is the same as that of K_{α_W} in $S^3 \setminus \operatorname{int} N(K) \approx S^1 \times D^2$.

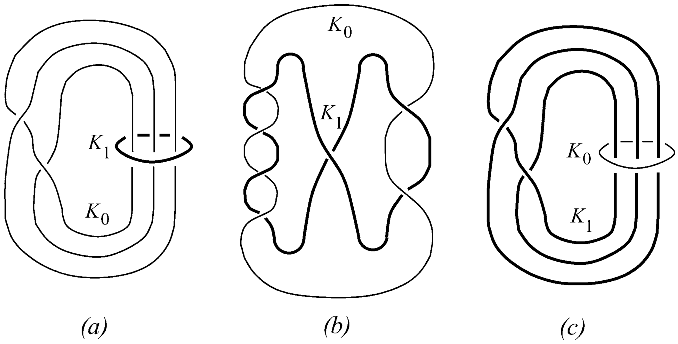


Fig. 18. The pretzel link $P(4, -1, -2) = K_0 \cup K_1 = K \cup K_{\alpha_W}$.

For convenience, denote $K = K_\mu$ and K_{α_W} in $W_K(\mu, \alpha_W) = S^3$ by K_0 and K_1, respectively; let $T_i = \partial_i W_K$ and $\mu_i \subset T_i$ be the meridian slope of K_i in T_i for $i = 0, 1$. Also, for $\{i, j\} = \{0, 1\}$, let D_i be a meridian disk of the solid torus $S^3 \setminus \operatorname{int} N(K_j) = X(\mu_i)$ which intersects K_i transversely, minimally, and coherently in 3 points (see Fig. 18); thus, K_i has wrapping and winding numbers equal to 3. Finally, let X denote the exterior of $K_0 \cup K_1$ in S^3, so that X and W_K are homeomorphic and hence X is hyperbolic.

We summarize our discussion so far in the following lemma.

Lemma 4.1. *X and W_K are homeomorphic with T_0, T_1 corresponding to $\partial_0 W_K$, $\partial_1 W_K$ and μ_0, μ_1 corresponding to μ, α_W, respectively. For $\{i, j\} = \{0, 1\}$, the solid torus $X(\mu_i)$ with meridian disk D_i contains two Möbius bands with boundary slopes s_i, t_i in $T_j = \partial X(\mu_i)$ satisfying $\Delta(s_i, \mu_j) = 1 = \Delta(t_i, \mu_j)$ and $\Delta(s_i, t_i) = 4$, each intersecting K_i transversely in one point, with ∂D_i the only slope in T_j satisfying $\Delta(s_i, \partial D_i) = 2 = \Delta(t_i, \partial D_i)$.*

Proof. We consider the case of $X(\mu_1)$, the other case being similar. The required Möbius bands in $X(\mu_1)$ are just the black and white surfaces B_1, B_2 in the projection of K_0 shown in Fig. 18(a); clearly, each of B_1 and B_2 intersects K_1 transversely in one point, with $\Delta(s_1, \mu_0) = 1 = \Delta(t_1, \mu_0)$ and $\Delta(s_1, t_1) = 4$. Notice that the once punctured Möbius bands $B_i' = B_i \cap X$ correspond to $P', Q' \subset W_K$ under the homeomorphism $X \approx W_K$, for which $\mu_1 \approx \alpha_W$. The uniqueness property of the slope of ∂D_1 in T_0 follows from the facts that $\Delta(s_1, t_1) = 4$ and $\Delta(s_1, \partial D_1) = 2 = \Delta(t_1, \partial D_1)$. □

The next two general results will be useful in the sequel. We remark that the parity rule of [5] applies to the transverse intersection of any two once-punctured surfaces in $(\mathcal{M}^3, T)$, whenever $\mathcal{M}^3$ is orientable and T is a torus boundary component of $\mathcal{M}^3$.

Lemma 4.2. *Let $\mathcal{M}^3$ be an irreducible* 3*-manifold with an incompressible torus boundary component T. Then,*

(i) *any Möbius band in $(\mathcal{M}^3, T)$ is essential,*
(ii) *any once-punctured Klein bottle $\mathcal{P}$ in $(\mathcal{M}^3, T)$ with nontrivial boundary slope is incompressible, and it either is essential or boundary compresses to a Möbius band B with $\Delta(\partial\mathcal{P}, \partial B) = 2$.*

Proof. Observe that if F is a once-punctured surface in $(\mathcal{M}^3, T)$, and D is a nontrivial boundary compression disk for F with $\partial D = \alpha \cup \beta$, where α, β are arcs such that $\partial\alpha = \partial\beta$, $\alpha \subset T$, and $\beta \subset F$, then the arc β must be negative in F. Hence F is nonorientable, and the boundary compression of F along D yields a once-punctured surface F' in $(\mathcal{M}^3, T)$ with $\Delta(\partial F', \partial F) = 2$ and $\operatorname{genus}(F') = \operatorname{genus}(F) - 1$.

Let $\mathcal{B}$ be a Möbius band in $(\mathcal{M}^3, T)$; if the circle $\partial\mathcal{B}$ is trivial in T then $\mathcal{M}^3$ has an RP^3 connected summand, which is not possible. If $\partial\mathcal{B}$ is nontrivial in T and $\mathcal{B}$ is boundary compressible, then $\mathcal{B}$ boundary compresses to a nontrivial disk in $(\mathcal{M}^3, T)$, contradicting the fact that T is incompressible. Thus (i) holds.

Suppose that $\mathcal{P}$ is a once-punctured Klein bottle in $(\mathcal{M}^3, T)$ with nontrivial boundary slope. If $\mathcal{P}$ compresses in $\mathcal{M}^3$ along some curve $\gamma \subset \mathcal{P}$ then γ must be orientation

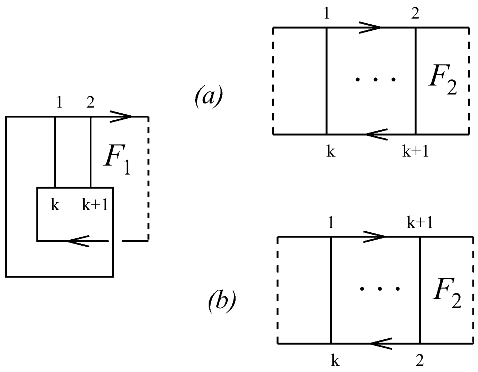


Fig. 19.

preserving in $\mathcal{P}$, hence either T compresses in $\mathcal{M}^3$ (if γ is a meridian) or $\mathcal{M}^3$ contains a closed projective plane (if γ is a longitude) and so $\mathcal{M}^3$ has an RP^3 connected summand, neither of which is the case. Therefore $\mathcal{P}$ is incompressible in $\mathcal{M}^3$.

Finally, if $\mathcal{P}$ is boundary compressible then $\mathcal{P}$ boundary compresses to a Möbius band B in $\mathcal{M}^3$ with $\Delta(\partial B, \partial\mathcal{P}) = 2$. Thus (ii) holds. □

Lemma 4.3. *Let $\mathcal{M}^3$ be a manifold with a torus boundary component T, and let F_1, F_2 be once-punctured surfaces in $(\mathcal{M}^3, T)$ intersecting transversely and minimally in essential graphs. Then no pair of arc components in $F_1 \cap F_2$ is parallel in both F_1 and F_2.*

Proof. The claim is trivial if $\Delta(\partial F_1, \partial F_2) \leqslant 2$, so we assume $\Delta(\partial F_1, \partial F_2) = 2n \geqslant 4$ and suppose a, b are two arc components of $F_1 \cap F_2$ which are parallel in both F_1, F_2. By the parity rule of [5] we may assume, without loss of generality, that a, b are negative in F_1 and positive in F_2. Since the graph $F_1 \cap F_2 \subset F_1$ is essential, we may also assume that a, b are adjacent in F_1, that is, there is no other component of $F_1 \cap F_2$ in the rectangle of parallelism cobounded by a, b in F_1.

Label the points $\partial F_1 \cap \partial F_2$ by $1, 2, \ldots, 2n$ in their order of appearance along ∂F_1, with some orientation. If $d = d(\partial F_1, \partial F_2)$, then the induced labelling on consecutive points of $\partial F_1 \cap \partial F_2$ along ∂F_2, with some orientation, is of the form $[1], [1+d], \ldots, [1+k\cdot d], \ldots$, where $[x]$ is the smallest nonnegative mod $2n$ reduction of x.

Since the graphs of intersection between F_1 and F_2 are essential, the possible graph pairs near a, b are the ones shown in Fig. 19. In Fig. 19(a), let R be the rectangle in F_2 cobounded by a, b, and suppose int R contains $s \geqslant 0$ arc components of $F_1 \cap F_2$; each such component is necessarily parallel to a, b in F_2. It follows that $[2] = [1 + (s+1)\varepsilon \cdot d]$ and $[k] = [k + 1 + (s+1)\varepsilon \cdot d]$ for some $\varepsilon = \pm 1$, hence $[2] = [0]$ and so $n = 1$, which is not the case. The situation in Fig. 19(b) is similar, so the claim follows. □

The general property of the graphs of intersection between once-punctured surfaces given in the lemma above immediately implies the bound $\Delta(\partial P, \partial Q) \leqslant 8$ between essential once-punctured Klein bottles; the proof is essentially the same as in [5, Lemma 12].

Corollary 4.4. *Let $\mathcal{M}^3$ be an irreducible 3-manifold with an incompressible torus boundary component T, and let $\mathcal{P}, \mathcal{Q}$ be essential once-punctured Klein bottles in $(\mathcal{M}^3, T)$. Then $\Delta(\partial\mathcal{P}, \partial\mathcal{Q}) \leqslant 8$.*

Proof. Isotope $\mathcal{P}$ so as to intersect $\mathcal{Q}$ transversely and minimally; then the graphs of intersection $G_{\mathcal{P}}, G_{\mathcal{Q}}$ are essential. By Lemma 4.3, no two arc components of $\mathcal{P} \cap \mathcal{Q}$ are parallel in both $\mathcal{P}$ and $\mathcal{Q}$. It follows that neither graph has parallel negative arcs: for a pair of parallel negative arcs in, say, $\mathcal{P}$, would give rise to a pair of positive arcs in $\mathcal{Q}$, which are necessarily parallel (see Section 2). Since any collection of at least 3 negative arcs in a once-punctured Klein bottle contains at least one parallel pair, both $G_{\mathcal{P}}$ and $G_{\mathcal{Q}}$ each contain at most two negative arcs, hence at most two positive arcs by the parity rule of [5]. Therefore $G_{\mathcal{P}}$ and $G_{\mathcal{Q}}$ each contains at most 4 arcs, so we must have $\Delta(\partial\mathcal{P}, \partial\mathcal{Q}) \leqslant 8$. □

Lemma 4.5. *Let $\mathcal{M}^3$ be an irreducible 3-manifold with an incompressible torus boundary component T, and let $\mathcal{B}$ be a Möbius band in $(\mathcal{M}^3, T)$. Suppose $\mathcal{P}$ is a once-punctured Klein bottle in $(\mathcal{M}^3, T)$ with nontrivial boundary slope. Then,*

(i) *if $\mathcal{P}$ is not essential, $\Delta(\partial\mathcal{P}, \partial\mathcal{B}) = 2$, and*
(ii) *if $\mathcal{P}$ is essential, $\Delta(\partial\mathcal{P}, \partial\mathcal{B}) = 0$ or 2 and $\mathcal{M}^3$ contains a closed nonorientable surface.*

Proof. Notice that any Möbius band in $(\mathcal{M}^3, T)$ is essential by Lemma 4.2. If $\mathcal{P}$ is not essential then, by Lemma 4.2, $\mathcal{P}$ boundary compresses to a Möbius band $\mathcal{B}'$ with $\Delta(\partial\mathcal{P}, \partial\mathcal{B}') = 2$. Now, if $\Delta(\partial\mathcal{B}, \partial\mathcal{B}') \neq 0$ then, after isotoping $\mathcal{B}$ so as to intersect $\mathcal{B}'$ transversely and minimally, any arc component of $\mathcal{B} \cap \mathcal{B}'$ must be essential in both $\mathcal{B}$ and $\mathcal{B}'$. But any essential arc in a Möbius band is negative, contradicting the parity rule of [5]. Therefore $\Delta(\partial\mathcal{B}, \partial\mathcal{B}') = 0$, so $\Delta(\partial\mathcal{P}, \partial\mathcal{B}) = 2$ and (i) follows.

We will assume that $\mathcal{P}$ is essential for the rest of the argument.

Isotope $\mathcal{P}$ so as to intersect $\mathcal{B}$ transversely and minimally, and let $G_{\mathcal{B}}, G_{\mathcal{P}}$ denote their graphs of intersection, necessarily essential. If $\Delta(\partial\mathcal{P}, \partial\mathcal{B}) \geqslant 4$ then $G_{\mathcal{B}}$ has at least two parallel negative arcs a, b; by the parity rule of [5], a, b are positive in $\mathcal{P}$ and hence parallel in $\mathcal{P}$ (see Section 2), contradicting Lemma 4.3. Therefore $\Delta(\partial\mathcal{P}, \partial\mathcal{B}) \leqslant 2$.

For the case $\Delta(\partial\mathcal{P}, \partial\mathcal{B}) = 2$, the possible graphs pairs $G_{\mathcal{B}}, G_{\mathcal{P}}$ are shown in Fig. 20, each consisting of one arc and no circle components. Let μ be any slope in T representing $d(\partial\mathcal{P}, \partial\mathcal{B}) = 1$, and let D be the disk $\mathcal{B} \setminus \operatorname{int} N(\mathcal{P})$, where $N(\mathcal{P})$ is a small regular neighborhood of $\mathcal{P}$ in $\mathcal{M}^3$. We will concentrate in the case of $G_{\mathcal{P}}$ shown at the top of Fig. 20, the other case being similar. If D^* is the extension of D to a disk properly embedded in $\mathcal{M}^3(\mu) \setminus \operatorname{int} N(\mathcal{P})$, then ∂D^* bounds a Klein bottle in $N(\mathcal{P})$ (see Fig. 21), hence $\mathcal{M}^3(\mu)$ contains a closed Klein bottle intersecting $K_\mu \subset \mathcal{M}^3(\mu)$ in $\Delta(\partial\mathcal{P}, \partial\mathcal{B}) = |\partial\mathcal{P} \cap \partial D^*| = 2$ points, so $\mathcal{M}^3$ contains a closed nonorientable surface.

Suppose now that $\Delta(\partial\mathcal{P}, \partial\mathcal{B}) = 0$; by (i), any such $\mathcal{P}$ is necessarily essential. Among all such pairs $\mathcal{P}, \mathcal{B}$, choose one which intersects transversely in the smallest number of components; in particular, $\partial\mathcal{P} \cap \partial\mathcal{B} = \emptyset$. Clearly, if $\mathcal{P}$ and $\mathcal{B}$ are disjoint then $\mathcal{M}^3$ contains a closed nonorientable surface. Otherwise, each component of $\mathcal{P} \cap \mathcal{B}$ is an essential circle which simultaneously preserves or reverses orientation in both $\mathcal{P}, \mathcal{B}$ (cf. Lemma 2.1).

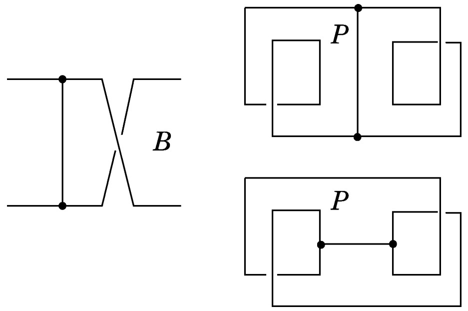


Fig. 20.

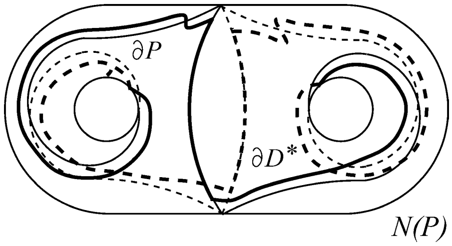


Fig. 21.

Let γ be an orientation preserving component of $\mathcal{P} \cap \mathcal{B}$. Then γ and $\partial\mathcal{B}$ cobound an annulus A in $\mathcal{B}$, and we can assume that $\mathcal{P} \cap \operatorname{int} A = \emptyset$ by choosing γ to be outermost in $\mathcal{B}$. If γ is parallel to $\partial\mathcal{P}$ in $\mathcal{P}$ then, by exchanging the annulus in $\mathcal{P}$ cobounded by $\gamma \cup \partial\mathcal{P}$ with A, we can construct a new once-punctured Klein bottle $\mathcal{P}'$ in $(\mathcal{M}^3, T)$ with the same boundary slope as $\mathcal{P}$ such that the pair $\mathcal{P}', \mathcal{B}$, after pushing $\mathcal{P}'$ off from $\mathcal{B}$, has fewer intersections than $\mathcal{P}, \mathcal{B}$, contradicting our minimal choice of $\mathcal{P}, \mathcal{B}$. If γ is a longitude in $\mathcal{P}$, and B_P is the Möbius band it bounds in $\mathcal{P}$, then the Möbius band $\mathcal{B}' = A \cup B_P$ can be isotoped to intersect $\mathcal{P}$ in a single common center component, again contradicting our minimal choice of $\mathcal{P}, \mathcal{B}$. If γ is a meridian of $\mathcal{P}$, then capping off the boundary components of $\mathcal{P} \setminus \operatorname{int} N(\gamma)$ with two disjoint parallel copies of A yields a pair of pants R in $(\mathcal{M}^3, T)$ with the same boundary slope as $\mathcal{P}$. Notice that R is necessarily incompressible, hence essential, and that each component in a minimal transverse intersection of $\mathcal{B}$ and R is orientation preserving and parallel to $\partial\mathcal{B}$ in $\mathcal{B}$ and to ∂R in R. If $B' \subset \mathcal{B}$ is the closure of the Möbius band component of $\mathcal{B} \setminus R$ then $\partial B' \subset R$ is parallel to some component of ∂R; hence it is possible to construct a new Möbius band $\mathcal{B}'$ in $(\mathcal{M}^3, T)$ which is disjoint from R, so $\mathcal{M}^3$ contains a closed nonorientable surface.

Finally, if $\mathcal{P} \cap \mathcal{B}$ consists of a single component c which is a center in both $\mathcal{P}, \mathcal{B}$, then the twice-punctured surface in $(\mathcal{M}^3, T)$ obtained from $\mathcal{P} \cup \mathcal{B}$ by exchanging the Möbius bands $N(c) \cap (\mathcal{P} \cup \mathcal{B})$ with an annulus component of $\partial N(c) \setminus (\mathcal{P} \cup \mathcal{B})$ is nonorientable, so again $\mathcal{M}^3$ contains a closed nonorientable surface, and (ii) follows. $\square$

We can now give a quick proof of Corollary 1.3 from the Introduction.

Proof of Corollary 1.3. Let K be a nontrivial 2-cable knot in S^3 with exterior X_K; then X_K contains an essential Möbius band, which is unique up to isotopy. By Lemmas 4.2 and 4.5, any once-punctured Klein bottle in X_K must boundary compress to such Möbius band. □

Our next result gives information about certain Dehn-fillings of X, the exterior of the pretzel link $P(4,-1,-2)$. Recall that $\partial X = T_0 \cup T_1$.

Lemma 4.6. *Let $\{i, j\} = \{0, 1\}$ and $r_i \neq \mu_i$ be a nontrivial slope in T_i. Then $(X(r_i), T_j)$ is irreducible and contains no Möbius bands if $\Delta(r_i, \mu_i) \neq 1, 3$.*

Proof. We consider the case $i = 0$, $j = 1$, the other case being similar. First observe that $X(r_0)$ cannot be a solid torus by [1], as K_0 is the 1-bridge 3-string braid $\sigma_1^{-1}\sigma_2$, which is conjugate to $W_2^{-2}W_3^1$ and so does not have the standard form $W_2^{e_1}W_3^{e_2}$, $e_1 = \pm 1$ (notation as in [1]). Therefore, as X is hyperbolic, $(X(r_0), T_1)$ is irreducible by Scharlemann's Theorem [6].

Suppose now that $X(r_0) = X \cup_{T_0} N(K_{r_0})$ contains a Möbius band B, with $\partial B \subset T_1$. Then B is essential in $X(r_0)$ by Lemma 4.2, hence the frontier of $N(B)$ is an essential separating annulus in $X(r_0)$ and, since X is hyperbolic, $\Delta(r_0, \mu_0) \leqslant 3$ by [4, Theorem 1.1].

Assume now that $\Delta(r_0, \mu_0) = 2$, and that B intersects $N(K_{r_0})$ transversely in the smallest possible number of meridian disks, say m. Let $B' = B \cap X$, an m-punctured Möbius band in X; notice $m \geqslant 1$ since X is hyperbolic. If B' compresses in X along some circle $\gamma \subset B'$, then either T_1 compresses in $X(r_0)$ (if γ is parallel in B to ∂B) or m is not minimal (if γ is trivial in B). Recall that a Möbius band boundary compresses, if at all, to a disk. Thus, if B' is boundary compressible in X, then either B' boundary compresses to a planar surface in X with one boundary component in T_1 at distance 2 from ∂B and m boundary components in T_0 of slope r_0, or to an annulus in X with the same boundary slope as ∂B in T_1 and boundary slope at distance 2 from r_0 in T_0 (which can only occur if $m = 1$), or $m > 1$ is not minimal. None of these situations can occur since T_1 is incompressible in $X(r_0)$, X is hyperbolic, and m has been chosen to be minimal. Therefore B' is essential.

By Lemma 4.1, there exist Möbius bands B_1, B_2 in $X(\mu_0)$ which intersect K_0 transversely in one point and have boundary slopes $\partial B_1, \partial B_2$ at distance 4 in T_1. Let $B_1' = B_1 \cap X$ and $B_2' = B_2 \cap X$. We can see that B_1' and B_2' are essential in X by an argument similar to the one used for B'. Without loss of generality, we may assume that $\Delta(\partial B_1, \partial B) \neq 0$. Since $\Delta(r_0, \mu_0) = 2$, B_1' can be extended to a once-punctured Klein bottle in $X(r_0)$ via a Möbius band in $N(K_{r_0})$, hence $\Delta(\partial B_1, \partial B) = 2$ by Lemma 4.5; that $\Delta(\partial B_2, \partial B) = 2$ as well follows now by a similar argument. Therefore, if D_0 is the meridian disk of the solid torus $X(\mu_0)$, then $\Delta(\partial B, \partial D_0) = 0$ by Lemma 4.1.

Isotope B_1' so as to intersect B' transversely and minimally, and let $G_{B_1'} = B_1' \cap B' \subset B_1'$, $G_{B'} = B_1' \cap B' \subset B'$ be their graphs of intersection; since B_1', B' are essential, these graphs have no inessential arcs. As $\Delta(\partial B_1, \partial B) = 2$, the graph $G_{B_1'}$ must be similar to one of the graphs shown in Fig. 22.

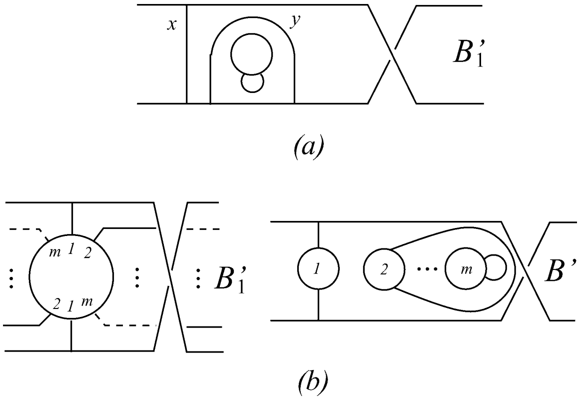


Fig. 22.

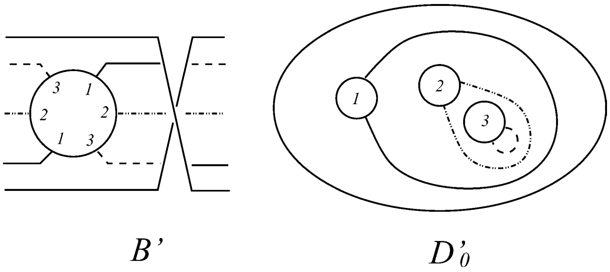


Fig. 23.

In Fig. 22(a) (where only one of the arcs x or y is present at a time) the graph $G_{B_1'}$ must have inessential arcs, while in Fig. 22(b) the graph $G_{B'}$ has inessential arcs if $m > 1$, neither of which can be the case. Therefore $m = 1$.

We now isotope B' so as to intersect $D_0' = D_0 \cap X$, an essential 3-punctured disk, transversely and minimally. As $\Delta(\partial B, \partial D_0) = 0$, ∂B and ∂D_0 are disjoint and so the essential graph $B' \cap D_0' \subset B'$ must be similar to the one shown in Fig. 23, which implies that the graph $B' \cap D_0' \subset D_0'$ has inessential arcs, contradicting the fact that B' is essential. Therefore, no such Möbius band $B \subset X(r_0)$ exists when $\Delta(r_0, \mu_0) = 2$. $\square$

We remark that, by Lemma 4.1, $X(r_0)$ indeed contains a Möbius band for each of the slopes $r_0 = s_1, t_1$, for which $\Delta(r_0, \mu_0) = 1$.

5. Klein pairs in 3-manifolds

In this section we consider embeddings of Klein pairs in arbitrary 3-manifolds. Our main result, Proposition 5.3, classifies those embeddings which yield essential Seifert Klein surfaces.

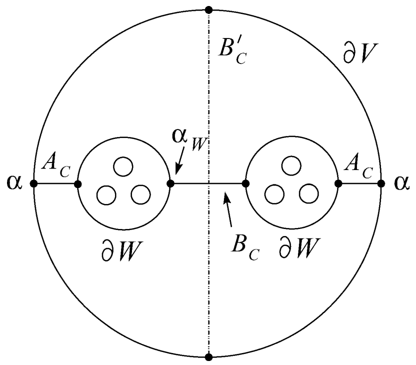


Fig. 24. The meridian disk of the solid torus V.

As in Section 4, let (V, K) be a Klein pair and W be a solid torus in V whose core is a 2-cable of the core of V and contains K as a 3-string closed braid. Let C denote the 2-cable space $V \setminus \operatorname{int} W$, and let $\alpha_W \subset \partial W$ and $\alpha \subset \partial V$ be the slopes of the essential annulus A_C which runs between the two boundary components of C; notice α_W is indeed the same slope as that of the once-punctured Möbius bands P', Q' in W_K constructed in Section 4, and that α_W is integral relative to W, while α runs twice along V. There are two embedded Möbius bands B_C, B'_C in C, where $\partial B_C \subset \partial W$ has slope α_W and $\partial B'_C \subset \partial V$ has slope α. See Fig. 24.

Construct now the lens space $(L_n, K) = (V, K) \cup_\partial U$, where U is a solid torus and $\pi_1(L_n) = \mathbb{Z}/n\mathbb{Z}$ for some integer $n \geqslant 0$. Let β be the slope in ∂V of the meridian disk of U. Denote by X_K, X_W the exteriors of K and W in L_n, and by V_K, W_K the exterior of K in V, W, respectively, and let μ be the meridian circle of K in V_K, W_K. ∂W_K has two components $\partial_0 W_K$ and $\partial_1 W_K = \partial W$, and V_K and W_K are irreducible with incompressible boundary.

Finally, let P, Q denote the standard Seifert Klein bottles in V_K (as defined by the end of Section 3) with boundary slopes at distance 4. We now analyze the essentiality (or lack of) of P, Q in $X_K \subset L_n$.

Lemma 5.1. *If $\Delta(\alpha, \beta) = 0$ then $X_K = S^1 \times D^2 \# RP^3$; hence the only essential punctured surface in X_K is a disk.*

Proof. Suppose that $\Delta(\alpha, \beta) = 0$, and let D be the meridian disk of U. Then D extends to a disk D_W in X_W via the essential annulus A_C in C, hence the slope of $\partial D_W \subset \partial W$ is α_W. If X is the exterior of the pretzel link $P(4, -1, -2)$ then, as $\alpha_W \approx \mu_1$ under the homeomorphism $W_K \approx X$ by Lemma 4.1 and $X(\mu_1)$ is a solid torus, it follows that $W_K \cup N(D_W) \approx X \cup N(D_X)$, where $\partial D_X = \mu_1$, and so $W_K \cup N(D_W) \subset X_K$ is a once-punctured solid torus. Also, since $\Delta(\alpha, \beta) = 0$, the Möbius band B'_C caps off to a closed projective plane in X_W. With this information and the fact that $W_K(\mu) = L_n = RP^3$ in this case, we can see that $X_K = S^1 \times D^2 \# RP^3$. $\square$

If N^3 is a 3-manifold with a torus boundary component T, a once-punctured Klein bottle $\mathcal{P}$ in (N^3, T) is said to be *unknotted* if its *exterior* $N^3 \setminus \operatorname{int} N(\mathcal{P})$ is a handlebody.

Lemma 5.2. *If $\Delta(\alpha, \beta) \neq 0$ then X_K is irreducible with incompressible boundary and contains no Möbius bands. In particular, the standard Seifert Klein bottles P, Q are essential in X_K, and if $\Delta(\alpha, \beta) = 1$ then P and Q are unknotted.*

Proof. That any once-punctured Klein bottle in X_K is essential will follow from Lemma 4.2 once we show that X_K is irreducible with incompressible boundary and contains no Möbius bands. We distinguish two cases.

Case 1. $\Delta(\alpha, \beta) \geqslant 2$.

Then $X_W = C \cup U$ is a Seifert fibered space over a disk with two singular fibers of indices 2 and $\Delta(\alpha, \beta) \geqslant 2$. Hence $X_K = W_K \cup_{\partial W} X_W$ is irreducible with incompressible boundary, and the torus ∂W is essential in X_K.

Suppose X_K contains a Möbius band B. As W_K is hyperbolic, B cannot lie in W_K. Isotope B so that B and ∂W intersect transversely in a minimal number of components. Then $B \cap \partial W$ consists of circles which are nontrivial in B and ∂W. Such circles must be orientation preserving in B and ∂W and hence each such circle is parallel to ∂B in B. It follows that the closure of the component of $B \setminus \partial W$ which contains ∂B is an essential annulus in W_K with one boundary component in each of $\partial_i W_K$, contradicting the fact that W_K is hyperbolic. Therefore X_K does not contain any Möbius bands.

Case 2. $\Delta(\alpha, \beta) = 1$.

Then $X_W = C \cup U$ is a solid torus (cf. case 1) which contains the Möbius band B_C properly embedded with boundary slope α_W; hence the boundary slope β_W of the meridian disk of X_W satisfies $\Delta(\alpha_W, \beta_W) = 2$. It follows that $X_K = W_K(\beta_W) \approx X(r)$, with X as in Section 4 and some slope r in T_1 with $\Delta(r, \mu_1) = 2$. Therefore $(X_K, \partial X_K)$ is irreducible and contains no Möbius bands by Lemma 4.6.

That P is unknotted in X_K can be seen as follows. Let $H = X_K \setminus \operatorname{int} N(P)$ denote the exterior of P in X_K, with $N(P)$ a small regular neighborhood of P. Let $B = P \cap X_W$ and $P' = P \cap W_K$, a Möbius band and a once-punctured Möbius band, respectively. Then $N(P) = N(B) \cup N(P')$ and $X_1 = X_W \setminus \operatorname{int} N(B)$ is a solid torus such that $A_B = \partial X_W \setminus \operatorname{int} N(\partial B)$ is an annulus in ∂X_1 which runs once around X_1. Hence, if $X_2 = W_K \setminus \operatorname{int} N(P')$, then $H = X_1 \cup_{A_B} X_2$ is homeomorphic to X_2. But as the slope of $\partial P'$ in $\partial_0 W_K$ runs once along $N(K)$, it is not hard to see that X_2 is homeomorphic to $W \setminus \operatorname{int} N(P') \approx X(\mu_0) \setminus \operatorname{int} N(B'_i)$, with X and B'_i as in the proof of Lemma 4.1 for some $i = 1, 2$, which can be easily identified as a genus 2 handlebody. That Q is unknotted too follows in a similar way. □

Notice that the case of knots in S^3 which are part of an *unknotted Klein pair* (that is, where the solid torus V in the Klein pair (V, K) is unknotted in S^3) corresponds to case 2 of Lemma 5.2. So, for example, the standard Seifert Klein bottles for the pretzel knot $P(-2,3,7)$ or its reflection are both unknotted. These two knots are the only such examples in S^3.

We are now ready to prove the main result of this section.

Proposition 5.3. *Let $\mathcal{M}^3$ be a 3-manifold which does not contain any closed Klein bottles, and let $K \subset \operatorname{int} \mathcal{M}^3$ be a knot with exterior $X_K = \mathcal{M}^3 \setminus \operatorname{int} N(K)$. Then K is spanned by two essential Seifert Klein bottles with boundary slopes at distance 4 iff K is part of some Klein pair $(V, K) \subset \mathcal{M}^3$ such that ∂V does not compress in $\mathcal{M}^3$ along the slope $\alpha \subset \partial V$. In such case, $(X_K, \partial N(K))$ does not contain any Möbius bands.*

Proof. Observe that if P, Q are essential in X_K then $\partial N(K)$ is necessarily incompressible in X_K, and that P, Q can be exchanged with a homeomorphic pair which intersects transversely, essentially, and minimally, via disk exchanges which correspond to isotopies if X_K is irreducible. Therefore, the only if part follows from the results of Section 2, Proposition 3.3, and Lemma 5.1.

Let (V, K) be a Klein pair in $\mathcal{M}^3$ with standard Seifert Klein bottles P, Q in V_K. If ∂V is incompressible in X_K then P and Q, which are essential in V_K, must be essential in X_K. That $(X_K, \partial N(K))$ does not contain any Möbius bands follows from an argument similar to that of case 1 of Lemma 5.2.

Suppose now that ∂V compresses in X_K along some disk $D \subset X_K \setminus \operatorname{int} V_K$ with boundary slope $\beta \neq \alpha$. Then $L = V \cup N(D)$ is a once-punctured lens space in $\mathcal{M}^3$ and P, Q are essential in $X_K \cap L$ by Lemma 5.2, hence P, Q are also essential in X_K; as $(X_K \cap L, \partial N(K))$ does not contain any Möbius bands by Lemma 5.2, neither does X_K. $\square$

We remark that the hypothesis that $\mathcal{M}^3$ does not contain any closed Klein bottle is not needed for the second part of the proof of Proposition 5.3.

6. Once-punctured Klein bottles

This section is devoted to the proof of Theorem 1.2; we will use the following notation. Let M^3 be an irreducible 3-manifold with an incompressible torus boundary component T, and let P, Q be two essential once-punctured Klein bottles in (M^3, T) with distinct boundary slopes r, s, respectively. We assume that P and Q have been isotoped so as to intersect transversely and minimally, so ∂P and ∂Q intersect transversely in $\Delta = \Delta(r, s)$ points. Notice that Δ is even, as P, Q each have only one boundary component. The next lemma classifies the graph pair G_P, G_Q for $\Delta = 2, 6, 8$.

Lemma 6.1. *If $\Delta = 2, 6,$ or 8, then the graph pair G_P, G_Q is combinatorially isomorphic to one the graph pairs of Fig. 25, and $P \cap Q$ has no circle components for $\Delta = 6, 8$.*

Proof. If $\Delta = 2$ then G_P, G_Q each must consist of one arc and, perhaps, some circle components. By the parity rule of [5], there are only two possible arrangements for the arc component of G_P and G_Q, as indicated in Fig. 25(a), (b). By the proof of Lemma 2.1,

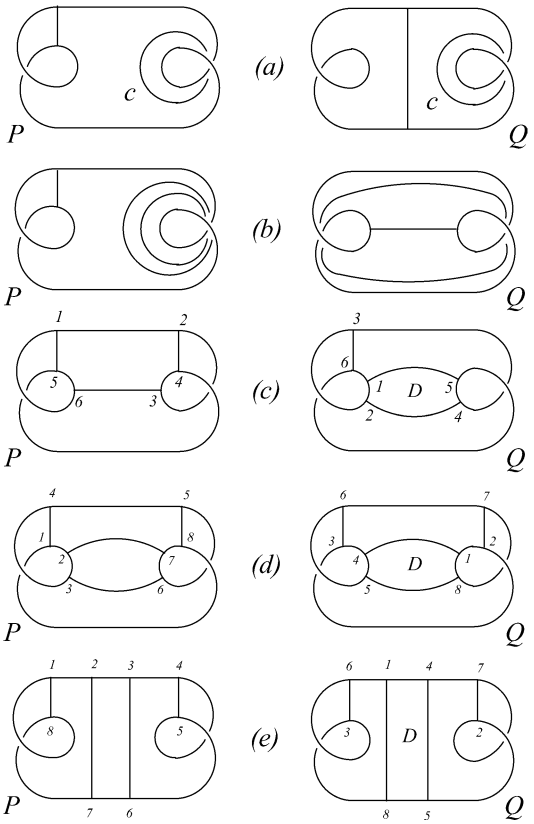


Fig. 25. The graph pairs G_P, G_Q for $\Delta = 2, 6, 8$.

a circle of intersection in (a) must be a longitude or center in both P and Q, while in (b) it must be a longitude in P and a meridian in Q.

If $\Delta = 6$ then each graph G_P, G_Q has exactly 3 arcs, and if μ is the unique slope representing $d(r,s) = 1$ then P, Q are Seifert Klein bottles for $K_\mu \subset M^3(\mu)$. Hence consecutive labelling of the points $\partial P \cap \partial Q$ along ∂P yields the same consecutive labelling along ∂Q. With this information and the parity rule of [5], it is not hard to see that the pair G_P, G_Q must be combinatorially isomorphic to the pair of Fig. 25(c).

The case $\Delta = 8$ and $d(r,s) = 1$ can be dealt with in a similar fashion; in this case the pair G_P, G_Q is combinatorially isomorphic to that of Fig. 25(d) (cf. [5, Proposition 13]).

For the case $\Delta = 8$ and $d(r,s) = 3$, consecutive labelling of the points $\partial P \cap \partial Q$ by $1,2,3,4,5,6,7,8$ along ∂P will result in the labelling $1,4,7,2,5,8,3,6$ of consecutive points along ∂Q, under some orientations of $\partial P, \partial Q$. It is now not hard to see that the pair G_P, G_Q must be combinatorially isomorphic to the pair of Fig. 25(e). $\square$

We now deal with the case $\Delta = 2$ in more detail.

Lemma 6.2. *If $\Delta = 2$ then $M^3(\mu)$ contains a closed nonorientable surface of genus 2 or 3 which intersects K_μ in 0 or 2 points, and M^3 contains a closed nonorientable surface.*

Proof. The situation here is similar to that of Section 2, so we only sketch the proof. Let μ be any slope in T realizing $d(r,s) = 1$ and $N(Q)$ a small regular neighborhood of Q in M^3. There are two cases to consider.

Case 1. Fig. 25(a).

Since each circle component of $P \cap Q$ is a center or longitude in both P or Q, either (i) $P \cap Q$ has no circle components, (ii) $P \cap Q$ contains one circle component c which is a center in P and Q, or (iii) $P \cap Q$ contains only longitudes. Notice that in case (iii) we can construct a closed Klein bottle in M^3 as in case 1 of Lemma 2.4.

In case (i), let $R = P \setminus \operatorname{int} N(Q)$, a Möbius band. Then R extends to a Möbius band $R^* \subset M^3(\mu) \setminus \operatorname{int} N(Q)$ such that $\alpha = \partial R^* \subset \partial N(Q)$ bounds a once-punctured Klein bottle F in $N(Q)$, hence $F \cup R^* \subset M^3(\mu)$ is a closed nonorientable surface of genus 3 which intersects K_μ in $\Delta = |\alpha \cap \partial P| = 2$ points. A representation of the circle α is shown in Fig. 26(a) (which, along with Fig. 26(b), is supposed to be superimposed on top of Fig. 2 in order to identify the circle ∂Q).

In case (ii), after a Möbius band exchange (as in Lemma 2.4), we can assume that $P \cap Q$ has exactly one circle component, which is c. Then $R = P \setminus \operatorname{int} N(Q)$ is an annulus, and $P \cap N(Q)$ has a Möbius band component S with $S \cap Q = c$, such that their corresponding extensions R^*, S^* satisfy $\partial R^* = \partial S^* \cup \alpha$, with α as in (i). Moreover, if F is the once-punctured Klein bottle in $N(Q)$ bounded by α, then $S \cap F$ is a common center circle c'. It follows that $F' = F \cup S \cup R$ is an immersed closed nonorientable surface of genus 3 in $M^3(\mu)$ with the center curve c' as its set of self-intersection points. Hence the surface obtained from F' by replacing $F' \cap N(c')$ with a component of $\partial N(c') \setminus F'$ is an embedded closed nonorientable surface in $M^3(\mu)$ of genus 3; again this surface intersects K_μ in $\Delta = |\alpha \cap \partial P| = 2$ points.

Case 2. Fig. 25(b).

If $P \cap Q$ has no circle components, then $R = P \setminus \operatorname{int} N(Q)$ is a Möbius band with extension $R^* \subset M^3(\mu) \setminus \operatorname{int} N(Q)$ such that $\partial R^* \subset N(Q)$ is a nontrivial separating curve β, which is represented in Fig. 26(b). If Q' is a once-punctured torus component of $\partial N(Q) \setminus \beta$, then $F = R^* \cup Q'$ is a closed nonorientable surface of genus 3 in $M^3(\mu)$ which intersects K_μ in $\Delta = |\beta \cap \partial P| = 2$ points.

Suppose now that $P \cap Q$ has circle components; each such circle must then be a longitude in P and a meridian in Q. We will assume that $P \cap Q$ has a single such circle component, the general case can be handled in a similar way. Then $P \cap N(Q)$ has an annulus component A with $A \cap Q$ the meridian circle of G_Q and boundary components the circles m_1, m_2 shown in Fig. 26(b), which are disjoint from ∂Q. Also, $P \setminus \operatorname{int} N(Q)$ consists of two components R_1, R_2, where R_1 is an annulus whose extension R_1^* has boundary components $m_i \cup \beta$ in $\partial N(Q)$, with β as above, and R_2 is a Möbius band whose extension R_2^* has boundary ∂R_2^* the circle m_j, for some $\{i, j\} = \{1, 2\}$. It follows that

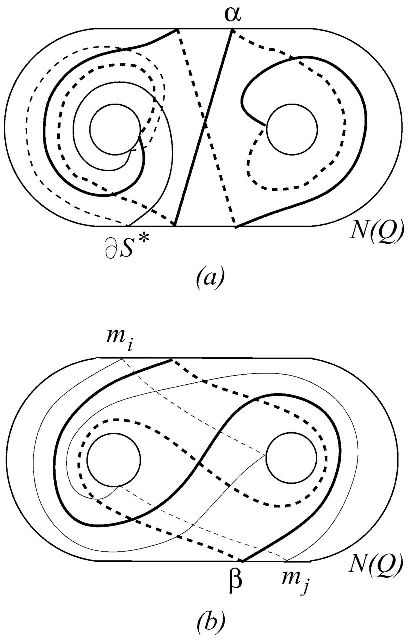


Fig. 26.

if Q' is some pair of pants component of $\partial N(Q) \setminus (m_1 \cup m_2 \cup \beta)$, then $Q' \cup R_1^* \cup R_2^*$ is a closed nonorientable surface of genus 3 in $M^3(\mu)$ which again intersects K_μ in $\Delta = |(m_1 \cup m_2 \cup \beta) \cap \partial P| = 2$ points.

Due to the nature of the curves $\alpha, \beta, m_1, m_2 \subset \partial N(Q)$ and the attached regions R^*, S^*, R_1^*, R_2^* considered above, if W^3 is a small regular neighborhood of $T \cup P \cup Q$ in M^3 then, in all cases, W^3 is irreducible, T is incompressible in W^3, and no slope in T bounds a Möbius band in W^3, so P and Q are necessarily essential in W^3 (cf. Lemma 5.2, case 1). Hence none of the above cases is topologically degenerate. $\square$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. The bound $\Delta \leqslant 8$ follows from Corollary 4.4, while (i) is the content of Lemma 6.2.

If $\Delta = 4$ then there is a unique slope μ realizing $d(r,s) = 1$; hence P, Q are essential Seifert Klein bottles for $K_\mu \subset M^3(\mu)$ and so (ii) follows from Lemma 2.4 and Proposition 5.3.

Let $N(P), N(Q)$ be regular neighborhoods of P, Q in M^3, respectively, which are sufficiently small so that their twice-punctured tori frontiers $T_P \subset \partial N(P)$, $T_Q \subset \partial N(Q)$ in M^3 intersect in essential graphs G_{T_P}, G_{T_Q}, each of which consists of two parallel copies of the double covers of G_P, G_Q, respectively.

With notation as in [2, §5–§6], it is not hard to see from Fig. 25(c)–(e) that the graph pair G_{T_P}, G_{T_Q} is combinatorially isomorphic to the pair $G(2,2,2,2,2)$, $G(4,2,0,0,2)$, respectively, if $\Delta = 6$, and to a pair of $G(4, 2, 2, 2, 2)$'s if $\Delta = 8$. By [2, §6], the labelling of these graphs is unique, depending only on the corresponding value of d in each case. Hence the graph pair G_{T_P}, G_{T_Q} is isomorphic to the pair $P(6)$, $P(8)_1$, and $P(8)_2$ of [2, §6] in the cases $\Delta = 6$, $\Delta = 8$ with $d = 1$, and $\Delta = 8$ with $d = 3$, respectively, so by [2, §11], M^3 is homeomorphic to $\mathcal{W}(2), \mathcal{W}(1)$, and $\mathcal{W}(-5)$, respectively, with T_P, T_Q essential in M^3. Thus P and Q are π_1-injective and (iii)–(v) hold. Since in (i) the manifold M^3 necessarily contains a closed nonorientable surface by Lemma 6.2, only (ii) and (iv) can be realized by knot exteriors in integral homology 3-spheres. □

We remark that in the cases where $\Delta = 6$ or 8, using techniques similar to those of Section 3, attaching a 2-handle along D^* to $N(P)$ in $M^3(\mu)$, where μ represents d and D is the disk face of G_Q shown in Fig. 25(c)–(e), yields a solid torus V containing an isotopic copy K of ∂P, such that $V_K = V \setminus \operatorname{int} N(K)$ is homeomorphic to $\mathcal{W}$, the exterior of the Whitehead link. A more elementary approach to Theorem 1.2(iii)–(v) which is independent from [2] could, in fact, be given along these lines.

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References

[1] J. Berge, The knots in $D^2 \times S^1$ which have nontrivial Dehn surgeries that yield $D^2 \times S^1$, Topology Appl. 38 (1) (1991) 1–19.
[2] C.McA. Gordon, Boundary slopes of punctured tori in 3-manifolds, Trans. Amer. Math. Soc. 350 (5) (1998) 1713–1790.
[3] C.McA. Gordon, J. Luecke, Toroidal and boundary-reducing Dehn fillings, Topology Appl. 93 (1) (1999) 77–90.
[4] C. Hayashi, K. Motegi, Dehn surgery on knots in solid tori creating essential annuli, Trans. Amer. Math. Soc. 349 (12) (1997) 4897–4930.
[5] K. Ichihara, M. Ohtouge, M. Teragaito, Boundary slopes of non-orientable Seifert surfaces for knots, Topology Appl. 122 (2002) 467–478.
[6] M. Scharlemann, Producing reducible 3-manifolds by surgery on a knot, Topology 29 (1990) 481–500.